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XIV.

SUPPLEMENTARY NOTE ON THE THEORY OF THE HORIZONTAL PHOTOHELIOGRAPH.

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Presented Dec. 12, 1877.

Referring to equation (14),* we remark that, if the photographs of the sun are centred upon the measuring engine with moderate care, it will seldom happen that either δy or δx amounts to so much as $3''$, while A will usually exceed $900''$. It is therefore evident that all three of these quantities cannot be accurately derived from conditional equations of the same form as (14), without using logarithms having at least six places of decimals. As the value of A is not required, we eliminate it in the following manner: In equation (14) put

$$\left.\begin{aligned} \sin(\varepsilon \mp \theta) &= a \\ n\cos(\varepsilon \mp \theta) &= b \\ \tfrac{1}{2} R\left[\sin^2(\varepsilon \mp \theta) + n\cos^2(\varepsilon \mp \theta)\right] - \frac{nA^2}{2R} &= n \end{aligned}\right\} \quad (58)$$

Then the resulting normal equations will be

$$\left.\begin{aligned} 0 &= [an] + [aa]\,\delta y + [ab]\,\delta x \\ 0 &= [bn] + [ab]\,\delta y + [bb]\,\delta x \end{aligned}\right\} \quad (59)$$

If m is the number of equidistant points at which R is measured, the expressions for the required auxiliaries are

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* Proceedings of the American Academy of Arts and Sciences, Vol. XII. (1876–77), p. 194.

$$\left.\begin{aligned}
[aa] &= \Sigma \sin^2 (\varepsilon \mp \theta) = \tfrac{1}{2}\, m \\
[ab] &= n\, \Sigma \sin (\varepsilon \mp \theta) \cos (\varepsilon \mp \theta) = 0 \\
[bb] &= n^2\, \Sigma \cos^2 (\varepsilon \mp \theta) = \tfrac{1}{2}\, m\, n^2 \\
[an] &= \Sigma \sin (\varepsilon \mp \theta) \left\{ \tfrac{1}{2} R [\sin^2 (\varepsilon \mp \theta) + n \cos^2 (\varepsilon \mp \theta)] - \frac{nA^2}{2\,R} \right\} \\
[bn] &= n \Sigma \cos (\varepsilon \mp \theta) \left\{ \tfrac{1}{2} R\, [\sin^2 (\varepsilon \mp \theta)] + n \cos^2 (\varepsilon \mp \theta)] - \frac{nA^2}{2\,R} \right\}
\end{aligned}\right\} \quad (60)$$

As the value of $[ab]$ comes out zero, we now evidently have

$$-\,\delta y = \frac{[an]}{[aa]} \qquad -\,\delta x = \frac{[bn]}{[bb]} \qquad (61)$$

If m is an even number, there will be $\frac{1}{2}\, m$ pairs of values of the angle $\varepsilon \mp \theta$, and in every case the two angles which compose the pair will differ from each other by 180°. If R and R' are the radii vectores of the two points composing a pair, and if the summation is extended only through $\frac{1}{2}\, m$ points, then putting

$$D = \tfrac{1}{2}\, (R - R')\, \left[1 + \frac{nA^2}{RR'} + (n - 1) \cos^3 (\varepsilon \mp \theta)\right] \qquad (62)$$

we have

$$\left.\begin{aligned}
-\,\delta y &= \frac{2}{m} \sum_{\epsilon \mp \theta = \pi}^{\epsilon \mp \theta = 0} \sin (\varepsilon \mp \theta)\, D \\
-\,\delta x &= \frac{2}{mn} \sum_{\epsilon \mp \theta = \pi}^{\epsilon \mp \theta = 0} \cos (\varepsilon \mp \theta)\, D
\end{aligned}\right\} \qquad (63)$$

To simplify these expressions still further, we remark that n is always very near unity, not becoming so small as 0.950 until a zenith distance of 85° is reached. Assuming $n = 1$ is equivalent to supposing the apparent sun to be replaced by an artificial one of perfectly circular outline, whose area is the same as, and whose centre coincides with the centre of gravity of, the apparent sun. As this assumption does not affect the values of δy and δx, we adopt it; and then equation (62) reduces to

$$D = (R - R') \qquad (64)$$

As the assumed artificial sun is circular in form, it has no minor axis; and therefore θ may be put equal to zero, and we have finally

$$\left.\begin{aligned} -\delta y &= \frac{2}{m}\sum_{\epsilon=\pi}^{\epsilon=0}\sin\varepsilon\,(R-R') \\ -\delta x &= \frac{2}{m}\sum_{\epsilon=\pi}^{\epsilon=0}\cos\varepsilon\,(R-R') \end{aligned}\right\} \qquad (65)$$

from which the values of δy and δx can be found by means of a table of natural sines and cosines, and a Crelle's multiplication table; no logarithms being required.

As it is now evident that the assumption of an elliptical outline for the apparent sun is unnecessary, we proceed to determine for δr a more accurate expression than equation (16). With this view, let *abcd*, Fig. 5, be the outline of the true sun, and *i* its centre, through which the vertical circle *ac* passes; and let *ebfd* be the outline of the apparent sun, and *k* its centre of gravity. Then, if *gh* is any diameter of the

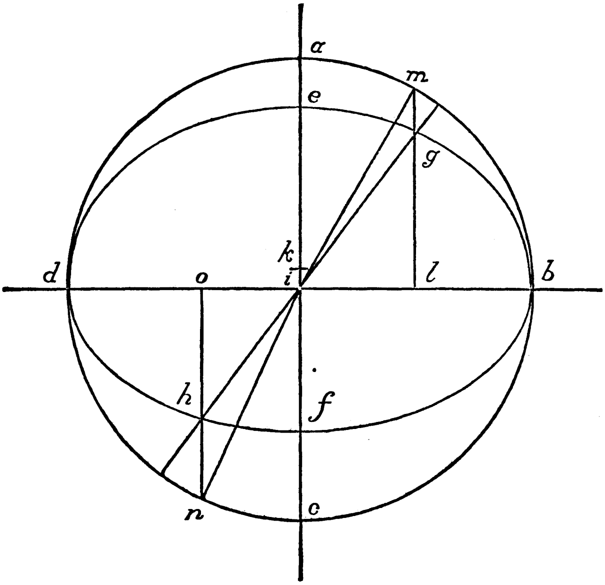

Fig. 5.

true sun, taking *ac* and *db* as a pair of rectangular axes, and putting $ai = s$, $gl = y_1$, $ho = y_2$, $aig = \theta$, $aim = \theta_1$, $ain = \theta_2$, we have

$$\left.\begin{aligned} y_1 &= s \cos \theta_1 - s \cos \theta_1 \frac{dr_1}{d\zeta_1} \\ y_2 &= s \cos \theta_2 - s \cos \theta_2 \frac{dr_2}{d\zeta_2} \end{aligned}\right\} \qquad (66)$$

in which $\frac{dr_1}{d\zeta_1}$ is the first derivative of the refraction, taken at a point midway between *m* and *l*; and $\frac{dr_2}{d\zeta_2}$ is the same derivative taken at a point midway between *n* and *o*. Adding these two equations, we obtain

$$y_1 + y_2 = s\,(\cos \theta_1 + \cos \theta_2) - s \cos \theta_1 \frac{dr_1}{d\zeta_1} - s \cos \theta_2 \frac{dr_2}{d\zeta_2} \qquad (67)$$

and, as the factors for converting the cosines of θ_1 and θ_2 into the cosine of θ can never differ much from unity, it will be quite accurate to write

$$y_1 + y_2 = s\,(\cos \theta_1 + \cos \theta_2) + s^2 \cos^2 \theta \frac{d^2r}{d\zeta^2} \qquad (68)$$

in which $\frac{d^2r}{d\zeta^2}$ is the second derivative of the refraction, taken at the centre of the true sun.

If we put $il = x_1$, and bear in mind that on account of refraction the horizontal diameter of the sun is contracted by a constant ratio, μ, then we have

$$x_1 = \mu s \sin \theta_1 \qquad (69)$$

and

$$\tan \theta = \frac{x_1}{y_1} = \tan \theta_1 \frac{\mu}{1 - \frac{dr_1}{d\zeta_1}} \qquad (70)$$

or

$$\tan \theta_1 = \frac{\tan \theta}{\mu} \left(1 - \frac{dr_1}{d\zeta_1}\right) \qquad (71)$$

Regarding θ_1 and $\frac{dr_1}{d\zeta_1}$ as variable, and differentiating, we get

$$\frac{d\,(\tan \theta_1)}{d\zeta} = - \frac{\tan \theta}{\mu} \frac{d^2r_1}{d\zeta^2_1} \qquad (72)$$

Assuming the well-known relations,

$$\frac{d\,(\cos\theta_1)}{d\theta_1} = -\sin\theta_1 \tag{73}$$

$$\frac{d\,(\tan\theta_1)}{d\theta_1} = \sec^2\theta_1 \tag{74}$$

dividing (73) by (74), multiplying the quotient by (72), and writing θ for θ_1, we have, with sufficient accuracy,

$$\frac{d\,(\cos\theta_1)}{d\zeta} = \left[\frac{\sin^2\theta\cos\theta}{\mu}\right]\frac{d^2r}{d\zeta^2} \tag{75}$$

The vertical distance between the centres of the ordinates y_1 and y_2 being very approximately $s\cos\theta$, we now have

$$s\,(\cos\theta_1 + \cos\theta_2) = -\left[\frac{\sin^2\theta\cos^2\theta}{\mu}\right]s^2\,\frac{d^2r}{d\zeta^2} \tag{76}$$

and therefore

$$y_1 + y_2 = \left[\cos^2\theta - \frac{\sin^2\theta\cos^2\theta}{\mu}\right]s^2\,\frac{d^2r}{d\zeta^2} \tag{77}$$

The value of μ is 0.99974. Consequently, it may be taken equal to unity without appreciable error, and then

$$y_1 + y_2 = \cos^4\theta\;s^2\,\frac{d^2r}{d\zeta^2} \tag{78}$$

As y_1 and y_2 are any pair of opposite ordinates of the curve which defines the outline of the apparent sun; and as δr is the distance *ik*, or, in other words, the ordinate of the centre of gravity of the apparent sun; we have, by reasoning identical with that employed in deducing equation (63),

$$\delta r = s^2\,\frac{d^2r}{d\zeta^2}\cdot\frac{2}{m}\sum_{\theta=+\frac{1}{2}\pi}^{\theta=-\frac{1}{2}\pi}\cos^4\theta \tag{79}$$

in which m is the number of points on the sun's limb at which ordinates have been computed. If we assume these points to be equidistant upon the apparent sun, and take $m = 12$, as in the measurements of the photographs, then

$$\delta r = 0.375\,s^2\,\frac{d^2r}{d\zeta^2} \tag{80}$$

To test the amount of inaccuracy introduced in equation (79), by writing θ for θ_1 in (75), and omitting μ in (78), the rigorous expression for δr, involving θ, θ_1, and μ, was formed; and, upon substituting in it the numerical values of these quantities for the extreme case where the sun has a zenith distance of 80°, the numerical coefficient in (80) became 0.374. This differs so little from the value before found as to lead to the conclusion, that equation (79) is probably as accurate as the refraction tables themselves.

We have thus found the values of δy, δx, and δr, and it only remains to deduce from them the expressions for $R \sin \varepsilon$ and $R \cos \varepsilon$; R and ε being respectively the radius vector and angle of the image of the true sun's centre, referred to the original system of polar co-ordinates given by the measuring engine. An inspection of Fig. 4 shows that δr is measured along the line $S'_{*}v'$, while δy and δx are measured along lines respectively parallel to, and perpendicular to, $S'_{*}Z$. Hence, as the angle ε is measured from a line at right angles to $S'_{*}Z$, and θ_s is the angle $ZS'_{*}S'$,

$$\left.\begin{aligned} R \sin \varepsilon &= \delta y + \delta r \cos \theta_s \\ R \cos \varepsilon &= \delta x \mp \delta r \sin \theta_s \end{aligned}\right\} \qquad (81)$$

in which δr must be regarded as essentially negative, θ_s must always be taken less than ninety degrees, and the double sign must be understood in the same way as in the equations (10). It is, perhaps, scarcely necessary to add, that these are the values of $R \sin \varepsilon$ and $R \cos \varepsilon$, which must be employed in the equations (18).

To facilitate the computation of δr from equation (80), it is desirable to have a table giving the numerical values of the second derivative of the refraction. To form such a table, we assume Bessel's expression for the refraction; namely,

$$r = \alpha \tan \zeta \qquad (82)$$

where r is the refraction, expressed in seconds of arc, and α is a quantity which varies slowly with the zenith distance, ζ. The first derivative of this expression is

$$\frac{dr}{d\zeta} = \frac{1}{\cos^2 \zeta}\left[\alpha + \tan \zeta \cos^2 \zeta \frac{d\alpha}{d\zeta}\right] \qquad (83)$$

which becomes

$$\frac{dr}{d\zeta} = \frac{1}{\cos^2 \zeta}\, \alpha' \qquad (84)$$

by putting

$$\alpha' = \alpha \text{ arc } 1'' + \tan\zeta \cos^2\zeta \frac{d\alpha}{d\zeta} \tag{85}$$

Differentiating (84), we get

$$\frac{d^2r}{d\zeta^2} = \frac{2\tan\zeta}{\cos^2\zeta}\left[\alpha' + \tfrac{1}{2}\cot\zeta \frac{d\alpha'}{d\zeta}\right] \tag{86}$$

which becomes

$$\frac{d^2r}{d\zeta^2} = \frac{2\tan\zeta}{\cos^2\zeta}\alpha'' \tag{87}$$

by putting

$$\alpha'' = \alpha' \text{ arc } 1'' + \tfrac{1}{2}\cot\zeta \frac{d\alpha'}{d\zeta} \tag{88}$$

The factor, arc 1″, is introduced in equations (85) and (88), because in (82) α is expressed in seconds of arc, while α' and α'' must be expressed in parts of radius. Bessel has given a table of the values of α', which may be found in his "Astronomische Untersuchungen," Vol. I. pp. 198 and 199; and also in Chauvenet's "Spherical and Practical Astronomy," Vol. II. pp. 572 and 573; but it must be noted that our α' is Bessel's α''. From the differences of consecutive values of α' given in that table, the values of $\frac{d\alpha'}{d\zeta}$ have been obtained, and then the values of α'', given below, were computed by means of equation (88).

The quantities in the following table correspond to a state of the atmosphere such that the barometer would stand at 29.597 inches, its attached thermometer at 32° F., and the external thermometer at 48°.75 F. The first and fourth columns of the table contain the argument. The second and fifth columns contain the values of log. α'', to the argument true zenith distance, computed by means of equation (88); but it must be carefully noted that, in using these logarithms, their characteristics must be diminished by 10. The third and sixth columns contain the values of δr, to the argument true zenith distance, computed by means of equation (80); and the seventh column, headed $\delta r'$, contains the values of δr to the argument apparent zenith distance, for all cases where they differ sensibly from the values given in the sixth column.

Strictly speaking, α'', δr, and $\delta r'$ are functions, not only of the zenith distance, but also of the density of the atmosphere; but so long as the temperature of the latter remains between + 30° and + 75° F., and its pressure between 29 and 30 inches of mercury, the values of δr and $\delta r'$ given in the table will not be in error more than $0''.01$, for any zenith distance less than 80°. For temperatures, pressures, or zenith distances, beyond these limits, if great accuracy is desired, the values of α'', δr, and $\delta r'$, must be multiplied by the factor $\beta^{A''} \gamma^{\lambda''}$, in which $\beta = BT$, and the values of A'', λ'', B, T, and γ, are to be taken from Bessel's table, cited above. To prevent misapprehension, it may be well to note that the values of A'' and λ'' given by Bessel apply rigorously only to the first derivative of the refraction; but their use with the second derivative gives rise to so little error that it does not seem worth while to recompute them especially for it.

ζ	Log. α''	δr	ζ	Log. α''	δr	$\delta r'$
°		''	°		''	''
0	1.1314	0.000	52	1.1238	0.003	
4	.1289	.000	54	.1229	.004	
8	.1286	.000	56	.1218	.004	
10	.1286	.000	58	.1205	.005	
12	.1286	.000	60	.1190	.006	
16	1.1286	0.000	62	1.1173	0.008	
20	.1285	.000	64	.1147	.010	
24	.1284	.001	66	.1118	.012	
28	.1283	.001	68	.1080	.016	
30	.1282	.001	70	.1034	.021	
32	1.1281	0.001	72	1.0972	0.029	0.029
34	.1280	.001	74	.0887	.040	.040
36	.1279	.001	76	.0776	.058	.059
38	.1278	.001	78	.0605	.089	.090
40	.1274	.001	80	.0310	.144	.147
42	1.1270	0.002	81	1.0107	0.188	0.193
44	.1265	.002	82	0.9858	.253	.261
46	.1259	.002	83	.9504	.349	.363
48	.1252	.002	84	.8991	.491	.517
50	.1246	.003	85	0.8431	0.747	0.809

WASHINGTON, May 30, 1877.